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LECTURES ON DIFFERENTIAL GAME[illegible]

L PONTRYAGIN

Lecture 1

AFOSR 71-1

We are going to talk about the problem of the pursuit of one controlled object by another controlled object. The most important feature of the problem lies in the fact that the future behavior of the object being pursued is not assumed to be known. In realizing the pursuit we must start from information concerning the state of the objects at a given time and the knowledge of the technical potentialities of these objects.

Here we denote the state of an object by x and assume that the controlled motions of the object can be described by the ordinary differential equation

$$\dot{x} = f(x,u) \tag{1}$$

where u is the control parameter.

x consists of two parts:

$$x = (x_1, x_2)$$

x_1 = geometrical position

x_2 = velocity.

Equation (1) gives the potentiality of the object by describing all possible motions of which the object is capable. In order to give the concrete motion of the object, we have to specify its initial state x_0 at some moment t_0 and then prescribe the values of the control u as a function of time.

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13. ABSTRACT

A series of 11 lectures on differential games presented at Stanford University in the fall of 1969 by the Ryssian Mathematician L. Pontryagin. The lectures are specifically devoted to the pursuit-evasion problem.

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In a pursuit problem one considers two objects x and y; the potentiality of the second object is described by the differential equation

$$\dot{y} = g(y,v) \quad (2)$$

Similar to (1), here v is a control parameter and y consists of two parts $y = (y_1,y_2)$, where y_1 denotes the geometrical position and y_2 denotes the velocity of the second object.

We assume that y moves in an arbitrary manner in accordance with (2) and that x aims to catch up with it in as short a time as possible, using all its technical capabilities, that is moving according to (1). The pursuit is considered to be completed at the instant when x and y coincide geometrically (i.e., $x_1 = y_1$).

The problem is: if at each instant of time one knows the states $x(t)$, $y(t)$ and $v(t)$, to prescribe the value $u(t)$ so that the pursuit is realized in the best way,

$$u = u(x,y,v) .$$

There is a different viewpoint to the problem: the problem of evasion, where $v(t)$ is chosen in order to avoid ending the game,

$$v = v(x,y,u) .$$

Now we reformulate these two problems, the pursuit problem and the evasion problem, as follows:

Define $z \triangleq (x,y)$, durect sum of x ard y. We call z the state of the game $\dot{z} = F(z,u,v)$ is the combination of (1) and (2).

The game ends when z hits the subspace $M = \{z \in R | x_1 = y_1\}$ of the phase space R.

The problem of pursuit is to find

$$u = u(z,v)$$ in order to end the game.

The problem of evasion is to find

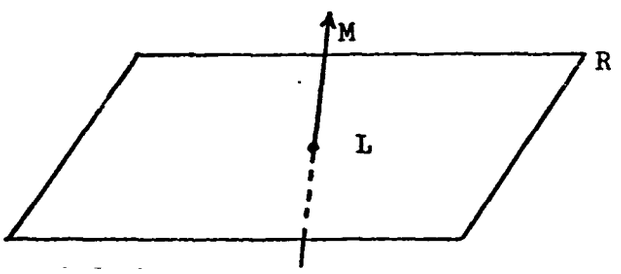
$$v = v(z,u)$$ in order to prolong the game.

Very many papers on the pursuit problem have been published, but not on the evasion problem. I will consider the evasion problem in the next lecture. Now I will only talk about the general formulation of the problems.

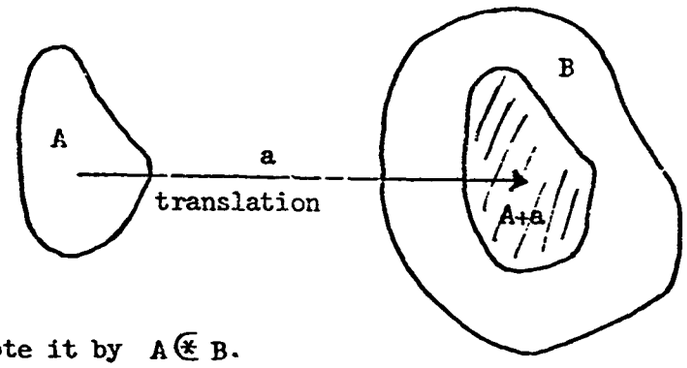
In order to obtain some concrete reults, we will restrict ourselves to linear problems. The differential equation of the linear differential game is written as follows:

$$\dot{z} = cz - u + v ,$$

where c is the $n \times n$ matrix, $u \in P$, $v \in Q$, and P, Q are convex and compact subsets of R. M is defined as before and let L be the subspace of R orthogonal to M; $L \perp M$ and $R = L \oplus M$.



If $A \subset L$ and $B \subset L$, we say that A is smaller than B iff $\exists$ a vector a such that $a + A \subset B$,



and we denote it by $A \Subset B$.

By π we denote the operator of orthogonal projection onto L. π is obviously a linear transformation. Let us consider the linear map $\pi e^{\tau c}$, where τ is any positive number, and define

$$P_\tau \triangleq \pi e^{\tau c} P$$

$$Q_\tau \triangleq \pi e^{\tau c} Q .$$

In order to get a positive solution to the pursuit game, the control parameter u has to have certain "superiority" over v. Similarly, in order to prolong the game indefinitely, v has to have certain "superiority" over u.

Proposition 1. Assume there exists $\tau_o > 0$, such that $\dim(P_\tau) = \dim(L) = \nu$ and $Q_\tau \Subset P_\tau$ for $0 < \tau < \tau_o$. Then there is a positive solution to the pursuit game in the sense that there is a subspace Ω with the property that if $z_o \in \Omega$, there is a finite $t(z_o)$ such that the game ends before $t = t(z_o)$.

Proposition 2. Assume there exist $\tau_o > 0$, $\mu > 1$ such that $\dim(Q_\tau) = \dim(L) = \nu \geq 2$ and $\mu P_\tau \subseteq Q_\tau$ for $\tau \in (0,\tau_o)$. Then for each $z_o \notin M$ $\exists v \ni \forall t$: $Z(t) \notin M$; in other words, the game never ends.

If the conditions in Proposition 2 are satisfied, $z(t)$ can be very close to M. We want to estimate the distance between $z(t)$ and M.

Define

$$\xi(t) = \text{distance between } z(t) \text{ and } M$$
$$= |\pi(z(t))|$$

and

$$\eta(t) = \text{distance between } z(t) \text{ and } L$$
$$= |z(t) - \pi(z(t))| .$$

We have, under certain conditions,

$$\xi(t) \geq \frac{c\xi^k(0)}{[1 + \eta(t)]^m} ,$$

where k and m are integers and c, k, and m all depend on the game itself.

Finally I want to point out that however complete the information concerning the state of the second object at a given instant of time may be, it is necessary to spend some finite amount of time in calculating and evaluating this information. Therefore the above formulation is not realistic. The way out of this difficulty is as follows: we can make the pursuit, not of y itself, but of the position in which it was found a short time beforehand. Thus $u(t)$ is chosen as a function of $x(t)$, $y(t-\Delta)$ and $v(t-\Delta)$, where Δ is a small positive number.

Lecture 2

Review: Consider the linear differential game

(1) $\dot{z} = cz - u + v$

$z \in R$ Euclidean space

$c: R \to R$

$u \in P$, $v \in Q$ $\Rightarrow$ convex, compact subsets of R

$\dim P$ and $\dim Q < \dim R$

M

$L \perp M$

$R = L \oplus M$

Projection $\pi: R \to L$

$P_\tau = \pi e^{\tau c} P$, $Q_\tau = \pi e^{\tau c} Q$

$\Rightarrow$ convex, compact

Let us consider the following

Example:

Question: If 1) E is a vector space, $\dim E = \nu \geq 2$, 2) $x \neq y \in E$ are the geometrical positions of the pursuer and the evader, 3) $x \neq y$ satisfy

(2) $\ddot{x} + \alpha\dot{x} = a$, $\quad |a| \leq \rho$,

(3) $\ddot{y} + \beta\dot{y} = b$ $\quad |b| \leq \sigma$,

where a and b are control vectors and $\alpha,\beta,\rho,\sigma > 0$, 4) the game ends when $x = y$, then what are the conditions for the pursuit problem and the evasion problem to have a positive solution?

Answer: If $\rho > \sigma$, the pursuit control has superiority over the evasion control; then the pursuit game has a positive solution. On the other hand, if $\rho < \sigma$, the pursuit control has inferiority over the evasion control; then the evasion game has a positive solution.

Proof: $(x,\dot{x})$ is the phase vector of x

$(y,\dot{y})$ is the phase vector of y

$(x,\dot{x},y,\dot{y})$ is the phase vector of the game.

The state variables $(x,\dot{x},y,\dot{y})$ can be reduced as follows. Let

$$z^1 = x - y$$

$$z^2 = \dot{x}$$

$$z^3 = \dot{y}$$

$$z = (z^1,z^2,z^3)$$

then

$$z^i \in E , \quad i = 1,2,3,$$

and

$$R = \{(z^1,z^2,z^3)\} ,$$

and

$$\dot{z}^1 = (x-y)^{\cdot} = \dot{x} - \dot{y} = z^2 - z^3$$

$$\dot{z}^2 = \ddot{x} = -\alpha\dot{x} + a = -\alpha z^2 + a$$

$$\dot{z}^3 = -\beta z^3 + b$$

and

$$u = (0,-a,0)$$

$$v = (0,0,b) .$$

Hence

$$P = \{(0,-a,0),\ |a| \leq \rho\}$$

$$Q = \{(0,0,b),\ |b| \leq \sigma\}$$

$$M = \{z \mid z^1 = 0\} = \{(0,z^2,z^3)\}\ \forall\ z^2,\ z^3 \in E$$

$$L = \{(z^1,0,0)\},\ \forall\ z^1 \in E$$

$$\pi(z^1,z^2,z^3) = (z^1,0,0) \triangleq z^1$$

$$c = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

In order to compute $\pi\, e^{tc} \begin{pmatrix} 0 \\ -a \\ 0 \end{pmatrix}$, we solve the following homogeneous differential equation

$$\dot{z} = cz\ ,$$

i.e.,

$$\dot{z}^1 = z^2 - z^3$$

$$\dot{z}^2 = -\alpha z^2$$

$$\dot{z}^3 = -\beta z^3\ .$$

Therefore,

$$z^3(t) = e^{-\beta t}\, z_o^3$$

$$z^2(t) = e^{-\alpha t}\, z_o^2$$

$$\dot{z}^1(t) = e^{-\alpha t}\, z_o^2 - e^{-\beta t}\, z_o^3$$

$$z^1(t) = z_o^1 + \frac{1 - e^{-\alpha t}}{\alpha}\, z_o^2 - \frac{1 - e^{-\beta t}}{\beta}\, z_o^3$$

$$e^{tc} = \left[1,\ \frac{1 - e^{-\alpha t}}{\alpha},\ -\frac{1 - e^{-\beta t}}{\beta}\right]$$

$$\pi e^{tc}\begin{pmatrix}0\\-a\\0\end{pmatrix} = \frac{1-e^{-\alpha\tau}}{\alpha}(-a)$$

Therefore

$$P_\tau = \{|x| \le \frac{1-e^{-\alpha\tau}}{\alpha}\rho\}$$

$$Q_\tau = \{|y| \le \frac{1-e^{-\beta\tau}}{\beta}\sigma\} .$$

For $\tau << 1$,

$$P_\tau = \{|x| \le \tau\rho + \cdots\}$$

$$Q_\tau = \{|y| \le \tau\sigma + \cdots\} .$$

Therefore if $\rho \ge \sigma$, $Q_\tau \overset{*}{\in} P_\tau$, then $\exists$ a solution to pursuit problem, and if $\sigma > \rho$, $\exists\ \mu > 1\ \ni \mu P_\tau \overset{*}{\in} Q_\tau$, then $\exists$ a solution to evasion problem.

Lecture 3

Review:

E = Euclidean vector space

$\dim(E) = \nu \geq 2$

$x, y \in I$

$\ddot{x} + \alpha\dot{x} = a$, $|a| \leq \rho$

$\ddot{y} + \beta\dot{y} = b$, $|b| \leq \sigma$ → positive numbers

$z^1 = x - y$

$z^2 = \dot{x}$

$z^3 = \dot{y}$

$$z = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}$$

$R = \{z = (z^1, z^2, z^3)\}$

$M = \{(0, z^2, z^3)\} = \{(z^2, z^3)\}$

$L = \{(z^1, 0, 0)\} = \{z^1\}$

$L \equiv F$

$P_\tau = \{|x| \leq \rho\tau + \cdots\}$

$Q_\tau = \{|y| \leq \sigma\tau + \cdots\}$

Let us consider the following two cases.

1) $\sigma > \rho$, then the evasion control has superiority over pursuit control.

Proof: $|z^1(t)|$ = distance between $z(t)$ and $M(t)$.

$|(z^2(t),z^3(t))|$ = distance between $z(t)$ and $L(t)$.

$$|z^1(t)| > \frac{c|z^1(0)|^2}{(1 + |(z^2(t),z^3(t))|)^m} ,$$

if $|z^1(0)| \leq \varepsilon$, where $\varepsilon > 0$ depends on the game, i.e.,

$$|x(t) - y(t)| > \frac{c|x(0) - y(0)|^2}{(1 + |(\dot{x}(t),\dot{y}(t))|)^M} ,$$

if $|x(0) - y(0)| \leq \varepsilon$.

Consider $\dot{z}^2 + \alpha z^2 = a(t)$. The solution is

$$z^2(t) = e^{-\alpha t} z_o^2 + \int_o^t e^{-\alpha\tau} a(t-\tau)d\tau .$$

As $t \to \infty$, $e^{-\alpha t} z_o^2 \to 0$, so

$$z^2(t) \doteq \int_o^t e^{-\alpha\tau} a(t-\tau)d\tau ,$$

if $t >> 1$. Hence

$$|\dot{x}(t)| \leq \frac{\rho}{\alpha} .$$

Similarly

$$|\dot{y}(t)| \leq \frac{\sigma}{\beta} .$$

Therefore, $|x(t) - y(t) > \gamma|x(0) - y(0)|^2$, for some $\gamma > 0$, when $|x(0) - y(0)| \leq \varepsilon$.

If $|x(0) - y(0)| = \varepsilon$, the above is true and we see that

$$\varepsilon > \gamma\varepsilon^2 .$$

If $|x(0)-y(0)| > \varepsilon$ and $|x(t)-y(t)| > \varepsilon$ for all t, then $|x(t)-y(t)| > \gamma\varepsilon^2$.

If $|x(0)-y(0)| > \varepsilon$ and $|x(t)-y(t)| = \varepsilon$ for some t, then, by the above result, we have $|x(t)-y(t) > \gamma\varepsilon^2$. Hence we get the following two cases

(1) If $|x(0)-y(0)| > \varepsilon$, then $|x(t)-y(t)| > \gamma\varepsilon^2$.

(2) If $|x(0)-y(0)| < \varepsilon$, then $|x(t)-y(t)| > \gamma|x(0)-y(0)|^2$.

In either case the game may last forever.

2) $\rho > \sigma$. Let us consider $\frac{\rho}{\alpha}$ and $\frac{\sigma}{\beta}$. As shown above, $\frac{\rho}{\alpha}$ characterizes the range of the velocity of the pursuer, and $\frac{\sigma}{\beta}$ characterizes the range of the velocity of the evader. Therefore,

$\frac{\rho}{\alpha} < \frac{\sigma}{\beta} \Longrightarrow$ the game may last forever,

$\frac{\rho}{\alpha} > \frac{\sigma}{\beta} \Longrightarrow$ the game may be ended.

My next lecture will be devoted to the evasion problem.

In order to show how P_τ and Q_τ are compared to judge whether the game may last forever or may be ended, we solve the following equation

$$(*) \qquad \begin{cases} \dot{z} = cz + v(t) - u(t) \\ z(0) = z_0 \end{cases}$$

and look at $\pi z(t)$. The solution to the corresponding homogeneous equation $\dot{z} = cz$ is $z(t) = e^{tc}c$. Let the solution to $(*)$ $z(t) = e^{tc}\,c(t)$, then

$$e^{tc}\dot{c} = v(t) - u(t)$$

$$\dot{c} = e^{-tc}(v(t) - u(t))$$

$$c(t) = z_o + \int_o^t e^{-sc}(v(s) - u(s))ds$$

$$z(t) = e^{tc} z_o + e^{tc} \int_o^t e^{-sc}(v(s) - u(s)ds$$

$$= e^{tc} z_o + \int_o^t (e^{\tau c} v(t-\tau) - e^{\tau c} u(t-\tau))d\tau$$

$$\pi z(t) = \pi e^{tc} z_c + \int_o^t (\pi e^{\tau c} v(t-\tau) - \pi e^{\tau c} u(t-\tau))d\tau$$

$$\pi e^{\tau c} v(t-\tau) \in Q_\tau$$

$$\pi e^{\tau c} u(t-\tau) \in P_\tau .$$

Lecture 4

Consider the linear transformation $\pi e^{\tau c}$: $R \to L$. Let us denote it by

$$g_\tau = \begin{pmatrix} g_1^1, g_2^1, \cdots, g_\mu^1 \\ \vdots \qquad\qquad \vdots \\ g_1^\nu, g_2^\nu, \cdots, g_\mu^\nu \end{pmatrix} . \tag{1}$$

Each of the elements is an analytic function of τ for small τ. It is well known that g_τ is "equivalent" to a matrix $G(\tau)$, in the sense that

$$g_\tau = A(\tau)G(\tau)B(\tau) , \tag{2}$$

where

$$G(\tau) = \begin{pmatrix} \tau^{\kappa_1} & & & & 0 \\ & \tau^{\kappa_2} & & & \\ & & \ddots & & \\ & & & \tau^{\kappa_\rho} & \\ & & & & 0 \\ 0 & & & & \ddots \; 0 \end{pmatrix} , \tag{3}$$

$$\kappa_1 \leq \kappa_2 \leq \cdots < \kappa_\rho ,$$

and $|A(0)| \neq 0$; $|B(0)| \neq 0$ (A, B nonsingular for small τ), where $|A(0)| = \det A(0)$. Let

$$C(\tau) = A^{-1}(\tau)$$

$$D(\tau) = B^{-1}(\tau) ,$$

then

(4) $$G(\tau) = C(\tau) g_\tau D(\tau)$$

$$|C(0)| \neq 0 \ , \quad |D(C)| = 0 \ .$$

Since each g_j^i is analytic, we can write

$$g_j^i = a\tau^K + b\tau^{K+1} + \cdots \ ,$$

where $a \neq 0$.

The interchange of two rows or two columns can be expressed as post-multiplication by a non-singular matrix, for instance,

$$\begin{bmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} g_2^1 & g_1^1 \\ g_2^2 & g_1^2 \end{pmatrix} .$$

Consider the operation

$$\begin{bmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{bmatrix} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g_1^1 , & g_2^1 + hg_1^1 \\ g_1^2 , & g_2^2 + hg_1^2 \end{bmatrix} .$$

Set $g_2^1 + hg_1^1 = 0$, then

$$h = - \frac{g_2^1}{g_1^1} .$$

Hence the matrix of the form $\begin{bmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{bmatrix}$ can be transformed into one of the form $\begin{bmatrix} g_1^1 & 0 \\ g_1^2 & g_2^2 \end{bmatrix}$ by multiplication by a non-singular matrix.

By repeated application of the preceding observations and similar ones about pre-multiplication, we see that g_τ can be transformed into

$$g_\tau = \begin{bmatrix} g_1^1 & 0 & 0 & \cdots & 0 \\ 0 & g_2^2 & 0 & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & & g_\rho^\rho & \vdots \\ 0 & & & 0 & 0 \end{bmatrix} .$$

Consider the operation

$$\begin{bmatrix} g_1^1 & 0 \\ 0 & g_2^2 \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} hg_1^1 & 0 \\ 0 & g_2^2 \end{bmatrix} = \begin{bmatrix} \tau^K & 0 \\ 0 & g_2^2 \end{bmatrix}$$

$$g_1^1 = \tau^K(a + b\tau + \cdots) .$$

Let $h = 1/a + b\tau + \cdots$.

Therefore, the above claim is proved:

$$g_\tau = A(\tau)G(\tau)B(\tau) .$$

Let $y = A(\tau)x$, then $p < \frac{|x|}{|y|} < q$, where p and q are two positive constants.

Theorem. Let $P = \{$all polynomials $f(\alpha,\beta)$ | in two variables such that deg $f = m$, with at least one of the coefficients equal to $+1$ or $-1\}$,

$$\Delta = \{(\alpha,\beta) | \alpha_1 \leq \alpha \leq \alpha_2,\ \beta_1 \leq \beta \leq \beta_2\} ,$$

and

$\Delta(\delta)$ = a square with dimension δ.

Then $\exists\ \delta > 0$ and $\sigma > 0$ $\ni$ for all $f \in P$ there exists $\Delta(\delta)$ (depending on f) such that

$$|f(\alpha,\beta)| > \sigma , \qquad \forall(\alpha,\beta) \in \Delta(\delta) , \text{ and } \forall\ f(\alpha,\beta) \in P .$$

Proof: Consider the compact subset $Q \subset P$ defined by

$$Q = \{f(\alpha,\beta) \in P | \text{the absolute value of every coefficient} \leq 1\} .$$

It can easily be shown that the conclusion is true for P replaced by Q. Then consider arbitrary $f(\alpha,\beta) \in P - Q$. Let

a = maximum of the absolute values of the coefficients of $f(\alpha,\beta)$,

then $a > 1$. Then

$$\frac{f(\alpha,\beta)}{a} \in Q .$$

Hence

$$\frac{|f(\alpha,\beta)|}{a} > \sigma , \qquad \forall(\alpha,\beta) \in \Delta(\delta)$$

Therefore

$$|f(\alpha,\beta)| > a\sigma > \sigma, \qquad \forall(\alpha,\beta) \in \Delta(\delta) .$$

This completes the proof.

Corollary. Let $P_a = \{f(\alpha,\beta) | \text{at least one of the coefficients is } +a \text{ or } -a, \deg f = m\}$.

Then $\exists\ \delta > 0$ and $\sigma > 0$ $\ni$ for all $f \in P$ there exists $\Delta(\delta)$ such that

$$|f(\alpha,\beta)| > a\sigma\ , \quad \forall (\alpha,\beta) \in \Delta(\delta)\ , \text{ and } \forall f(\alpha,\beta) \in P_a\ .$$

Lecture 5

Theorem. Consider two polynomials

$$x(t) = a + a_1 t + a_2 t^2 + \cdots + a_k t^k + \alpha t^k$$

$$y(t) = b + b_1 t + b_2 t^2 + \cdots + b_\ell t^\ell + \beta t^\ell \ .$$

Suppose

$$|a_i| \le K \ , \qquad |a_k + \alpha| \le K$$

$$|b_i| \le K \ , \qquad |b_\ell + \beta| \le K \ ,$$

where K is a positive number. Then there exists $\delta > 0$ such that

$$|x(t)| + |y(t)| > \frac{C(|a|^\ell + |b|^k)}{K^{k+\ell-1}} \ ,$$

$$\forall\ (\alpha,\beta) \in \Delta(\delta) \quad \text{and} \quad t \in [0,1] \ .$$

Remark. $(x(t),y(t))$, $t \in [0,1]$ defines a curve in the (x,y)-plane. $|x(t)| + |y(t)|$ is an estimate of the distance between the point $(x(t), y(t))$ and the origin $(0,0)$.

Proof. Let us denote the discriminant of the two polynomials x and y by

$$D = R(\alpha,\beta) \ .$$

From algebra we know that if $\mathcal{D} = 0$, (x,y) passes through $(0,0)$, if $\mathcal{D} \neq 0$, (x,y) does not pass $(0,0)$. We now compute $\mathcal{D}(\alpha,\beta)$.

Multiplying the first polynomial by $1,t,t^2,\ldots,t^{\ell-1}$ and multiplying the second polynomial by $1,t,t^2,\ldots,t^{k-1}$, we have

$$a + a_1 t + \cdots + a_k t^k + \alpha t^k + 0 \cdot t^{k+1} + \cdots + 0 \cdot t^{k+\ell-1} = x$$

$$0 + at + a_1 t^2 + \cdots + a_k t^{k+1} + \alpha t^{k+1} + \cdots + 0 \cdot t^{k+\ell-1} = xt$$

$$0 + \cdots + 0 + at^{\ell-1} + \cdots + a_k t^{k+\ell-1} + \alpha t^{k+\ell-1} = xt^{\ell-1}$$

and

$$b + b_1 t + \cdots + b_\ell t^\ell + \beta t^\ell + \cdots + 0 \cdot t^{k+\ell-1} = y$$

$$0 + bt + \cdots + b_\ell t^{\ell+1} + \beta t^{\ell+1} + \cdots + 0 \cdot t^{k+\ell-1} = yt$$

$$0 + \cdots + 0 + bt^{k-1} + \cdots + b_\ell t^{k+\ell-1} + \beta t^{k+\ell-1} = yt^{k-1} .$$

We consider the above equations as a system of simultaneous algebraic equations for the variables $1,t,t^2,\ldots,t^{k+\ell-1}$. From elementary algebra we see that

$$1 = \frac{\hat{\mathcal{D}}}{\mathcal{D}} ,$$

where $\mathcal{D}$ is the coefficient matrix and $\hat{\mathcal{D}}$ is $\mathcal{D}$ with the first column replaced by $(x,xt,\ldots,xt^{\ell-1}, y,yt,\ldots,yt^{k-1})$.

$$\mathfrak{D}(\alpha,\beta) = a^{\ell}\beta^{k} \pm b^{k}\alpha^{\ell} + \cdots$$

$$\mathfrak{D}(\alpha,\beta) \in P^{n}_{|a|^{\ell}}$$

$$\mathfrak{D}(\alpha,\beta) \in P^{n}_{|b|^{k}} .$$

Using the lemma we mentioned in the last lecture, we obtain

$$|\mathfrak{D}(\alpha,\beta)| > \sigma|a|^{\ell} , \qquad \text{if } (\alpha,\beta) \in \text{a certain square } \Delta'(\delta)$$

$$|\mathfrak{D}(\alpha,\beta)| > \sigma|b|^{k} , \qquad \text{if } (\alpha,\beta) \in \text{a certain square } \Delta''(\delta) .$$

Hence

$$|\mathfrak{D}(\alpha,\beta)| > (|a|^{\ell} + |b|^{k})\frac{\sigma}{2} , \quad \text{if } (\alpha,\beta) \in \Delta(\delta) ,$$

where $\Delta(\delta) = \Delta'(\delta)$ or $\Delta''(\delta)$ according to whether $|a|^{\ell} \geq |b|^{k}$ or $|a|^{\ell} \leq |b|^{k}$. Then we have

$$\frac{\hat{\mathfrak{D}}}{(|a|^{\ell} + |b|^{k})\frac{\sigma}{2}} \geq 1.$$

Since $|t| \leq 1$, we have $|xt^{i}| \leq |x|$, $|yt^{j}| \leq |y|$. On the other hand, any minor of order $k+\ell-1$ of $\mathfrak{D}$ is bounded by

$$(k+\ell-1)!K^{k+\ell-1} ,$$

then

$$\hat{\mathfrak{D}} \leq (\ell|x| + k|y|)(k+\ell-1)!K^{k+\ell-1}$$

which implies

$$\hat{\mathfrak{d}} \leq \gamma(|x| + |y|)K^{k+\ell-1} .$$

Consequently,

$$|x| + |y| > \frac{C(|a|^{\ell} + |b|^{k})}{K^{k+\ell-1}}$$

for some constant $C > 0$ as desired.

Lecture 6

Here we only consider the case where g_τ is a square $\nu \times \nu$ matrix

$$g_\tau = A(\tau)G(\tau)B(\tau), \quad \det A(0) \neq 0 , \quad \det B(0) \neq 0$$

$$G(\tau) = \begin{pmatrix} \tau^{K_1} & & & 0 \\ & \tau^{K_2} & & \\ & & \ddots & \\ 0 & & & \tau^{K_\nu} \end{pmatrix} .$$

A simple calculation yields

$$\int_0^t A(\tau)G(\tau)d\tau = \hat{A}(t)\hat{G}(t) ,$$

where $\det \hat{A}(0) \neq 0$ and

$$\hat{G}(t) = \begin{pmatrix} t^{K_1+1} & & & 0 \\ & t^{K_2+1} & & \\ & & \ddots & \\ 0 & & & t^{K_\nu+1} \end{pmatrix} .$$

Consider a vector $\varphi(\tau)$ with components φ_i satisfying $|\varphi_i| \leq c^\tau$, $c > 0$. Again a simple calculation yields

$$\int_0^t A(\tau)G(\tau)\varphi(\tau)d\tau = \hat{A}(t)\hat{G}(t)\hat{\varphi}(t) ,$$

where $|\hat{\varphi}_i(t)| \leq ct$, if $0 \leq t \leq \rho$, $\rho > 0$, where ρ depends only on $A(\tau)$.

Now let us continue the discussion of the differential game. We assume $z(t_c)$ is very near to M. Our purpose is to prevent the ending of the game in the time interval $[t_o,t_1]$. Let us assume, without loss of generality, $t_o = 0$ and $t_1 = \theta$.

We now rewrite the equation of the game as

$$\dot{z} = cz + v' - u' , \qquad u' \in P' \text{ and } v' \in Q',$$

in order to save the notations v and u for later use.

Recall that a condition for the evasion problem to have a solution is that $\dim(\pi e^{\tau c} Q') = \nu$. We make this assumption here. Hence,

$$\dim(Q') \geq \nu .$$

We will consider only the case $\dim(Q') = \nu$ to illustrate the idea of proof.

Obviously, $\exists$ affine subspace u' such that $P' \subset U'$, $\dim P' = \dim U'$. Then $\exists\, u_o \in U'$ and a linear subspace $U \subset L$

$$U' = u_o + U .$$

If we choose $u_o \in \mathrm{Int}(P')$, then $P' = u_o + P$ and $0 \in P$. Similarly, $\exists$ affine subspace $V' \ni Q' \subset V'$ and $\exists\, v_o \in V'$ and a linear subspace $V \subset L \ni V' = v_o + V$ and $Q' = v_o + Q$.

Define

$$f_\tau(u) = \pi e^{\tau c} u, \qquad \forall\, u \in U ,$$

$$g_\tau(v) = \pi e^{\tau c} v, \qquad \forall\, v \in V .$$

Notice that g_c is non-degenerate (non-singular) mapping: $V \to L$, by our assumption

$$\dim(Q') = \nu = \dim(L) , \qquad \text{for } \tau > 0 ,$$

$$\mu\pi\, e^{\tau c} P' \Subset \pi\, e^{\tau c} Q' \Longrightarrow \mu f_\tau(P) \Subset g_\tau(Q) .$$

Define

$$h_\tau = g_\tau^{-1} f_\tau , \qquad \tau > 0 .$$

Hence

$$\mu h_\tau(P) \Subset Q .$$

Notice that each element of g_τ^{-1} can be expressed in the form

$$a\tau^K + b\tau^{K+1} + \cdots ,$$

where K may be negative. Therefore h_τ might have a similar form. If this is the case, $\exists\, u \in U \ni$

$$|h_\tau(u)| \to \infty, \qquad \text{as } \tau \to 0 .$$

But $\mu h_\tau(P) \Subset Q$ and Q is compact, so we obtain a contradiction and thus we know that $h_\tau(u)$ is an analytic function of τ even at $\tau = 0$ and h_τ has a definite limit at $\tau = 0$. Hence it is possible to choose $v_o \ni \mu h_o(P) \subset Q$. But $\mu h_o(P) \subset Q \Longrightarrow h_o(P) \subset \frac{Q}{\mu} \subset Q$, so we can find $w \in V$ with components $|w_j| \leq \gamma$, $\gamma > 0$ such that

$$\frac{Q}{\mu} + w \subset Q .$$

Hence

$$h_o(P) + w \subset Q .$$

Let us assume we have a pursuit control. Consider the evasion control $v(t) = h_o(u(t)) + w$.

First,however, it is necessary to do the following computation

$$g_\tau(v(t-\tau)) - f_\tau(u(t-\tau)) = g_\tau(v(t-\tau) - h_\tau(u(t-\tau)))$$

$$= g_\tau(h_o(u(t-\tau)) + w - h_\tau(u(t-\tau)))$$

$$= g_\tau(w - (h_\tau - h_o)(u(t-\tau)))$$

$$= A(\tau)G(\tau)B(\tau)(w - (h_\tau - h_o)(u(t-\tau)))$$

$$= A(\tau)G(\tau)(w(\tau) + \varphi(t,\tau))$$

$$\int_o^t A(\tau)G(\tau)(w(\tau) + \varphi(t,\tau))d\tau = \hat{A}(t)\hat{G}(t)(w(t) + \hat{\varphi}(t)) \ .$$

This will be used in the future.

Lecture 7

As shown in the last lecture,

$$\int_0^t (g_\tau(v(t-\tau)) - f_\tau(u(t-\tau)))d\tau = \hat{A}(t)\hat{C}(t)(w + \hat{\varphi}(t))$$

$$\pi z(t) = \pi e^{\tau c} z_o + \int_0^t (\pi e^{\tau c} v'(t-\tau) - \pi e^{\tau c} u'(t-\tau))d\tau$$

$$= \pi e^{\tau c} z_o + \int_0^t (\pi e^{\tau c}(v_o+v(t-\tau)) - \pi e^{\tau c}(u_c+u(t-\tau))d\tau$$

$$= \pi e^{\tau c} z_o + \int_0^t \pi e^{\tau c}(v_o-u_o)d\tau + \int_0^t (g_\tau(v(t-\tau)) - f_\tau(u(t-\tau)))d\tau$$

$$= \pi e^{\tau c} z_o + \int_0^t \pi e^{\tau c}(v_o-u_o)d\tau + \hat{A}(t)\hat{G}(t)(w+\hat{\varphi}(t)) .$$

For $t \ll 1$, $\hat{A}(t)$ is non-singular, so

$$\hat{A}^{-1}(t)\pi z(t) = \omega(t) + \hat{G}(t)(w+\hat{\varphi}(t)) , \tag{6}$$

where

$$\omega(t) = \hat{A}^{-1}(t)(\pi e^{\tau c} z_o + \int_0^t \pi e^{\tau c}(v_o-u_o)d\tau).$$

Note that $\omega(t)$ is a linear function of z_o. Since $\hat{A}(0)$ is non-singular, we can take a system of coordinates in which $\hat{A}(0) = 1$, then we have $\omega(0) = \pi z_o$. Observe that (6) contains ν scalar equations. Any one of them can be expressed in the form

$$x = f(t) + t^k v(t) + t^k(\alpha_o + \psi(t))$$

($v(t)$: polynomial of degree k)

$$= f(t) + t^k(\alpha_o + \alpha_1(t))$$

$$= f(t) + t^k \alpha ,$$

where

$$\alpha = \alpha_o + \alpha_1 , \qquad \alpha_1 = \gamma(t) + \psi(t) ,$$

$$f(t) = a + a_1 t + \cdots + a_k t^k .$$

Consider the mapping $z \to (\xi,\eta)$, where $\xi = |\pi(z)|$ and $\eta = |z-\pi(z)|$. In particular, $z_o \to (\xi_o,\eta_o)$. Since we assume z_o is very close to M, we have $\xi_o \leq 1$. Hence we obtain the following inequality for the coefficients a_i ,

$$|a_i| \leq \rho(1 + \eta_o) , \qquad i = 0,\ldots,k$$

for some $\rho \geq 0$. And analogously we get

$$|\alpha_1(t)| \leq t\rho(1 + \eta_o)$$

$\rho \geq 0$, the same ρ as above. It is necessary to assume

$$t \leq \frac{\theta_o}{1 + \eta_o} , \qquad \theta_o > 0 .$$

If θ_o is small enough, then $\alpha_1(t)$ small enough. θ_o will be chosen later. Now θ_o is regarded as a fixed number. Let $\theta = \frac{\theta_o}{1 + \eta_o}$. And the coefficient of the last term in the above polynomial of x satisfies

$$|a_k + \alpha| \leq \rho(1 + \eta_o) .$$

Now let us introduce some new terminologies,

$$z \in L , \qquad z = (z^1,z^2,\ldots,z^\nu) .$$

It is easy to see that $\exists$ p, q $\ni$

$$|z| > p|z^i| , \qquad \forall\, i = 1,\ldots,\nu ,$$

and

$$|z^i| > q|z| , \qquad \text{for at least one } i$$

Consider the instant $t = \theta$,

$$x = f(\theta) + \alpha\theta^k .$$

We consider two cases:

1) $f(\theta) \geq 0$, $\quad \frac{\gamma}{2} \leq \alpha \leq \gamma$, $\quad |x| > \frac{\gamma}{2}\theta^k$,

2) $f(\theta) \leq 0$, $\quad -\gamma \leq \alpha \leq -\frac{\gamma}{2}$, $\quad |x| > \frac{\gamma}{2}\theta^k$.

We have different controls for these two cases. Now we have

$$\xi(\theta) > \frac{\gamma}{2}\theta^k$$

and

$$\xi(\theta) > \frac{\lambda}{(1+\eta_o)^k}$$

We now consider the second equation

$$y = g(t) + t^{\ell}\beta , \qquad \ell \leq k .$$

Applying the theorem we mentioned in Lecture 5, we have

1) $\frac{\gamma}{2} \leq \alpha \leq \gamma$, $\quad -\gamma \leq \beta \leq \gamma \rightarrow \Delta'$

2) $-\gamma \leq \alpha \leq -\frac{\gamma}{2}$, $\quad -\gamma \leq \beta \leq \gamma \rightarrow \Delta''$.

Therefore δ', $\delta'' > 0$ ∋

$$|x| + |y| > \frac{c_1(|a|^{\ell} + |b|^{k})}{K^{k+\ell-1}} .$$

Let $\delta = \min(\delta', \delta'')$.

$$|x| + |y| > \frac{c_1(|a|^{\ell} + |b|^{k})}{K^{k+\ell-1}} ,$$

where K is an upper bound of the absolute value of the coefficients of the x,y polynomials.

$$\underbrace{|x| + |y|}_{\text{(an estimate of } \xi)} \geq \frac{c_1(|a|^{\ell} + |b|^{k})}{(\rho(1 + \eta_o))^{2k-1}} .$$

Hence

$$\xi(t) > \frac{c_3 \xi_o^{k}}{(1 + \eta_o)^{2k-1}} .$$

Lecture 8

At the end of the last lecture we got the two following inequalities:

(1) $$\xi(\theta) > \frac{\lambda}{(1 + \eta_o)^k}$$

(2) $$\xi(t) > \frac{c_o \xi_o^k}{(1 + \eta_o)^{2k-1}} .$$

But these inequalities are not interesting since they relate $\xi(\theta)$ or $\xi(t)$ with η_o and not with $\eta(t)$.

To remedy to this, we will use the following relation:

$$1 + \eta(t) > S(1 + \eta_o) \qquad \text{for some } S.$$

This can easily be shown by using the fact that for t being small $z(t) - z(0)$ is bounded. Then

(3) $$\xi(\theta) > \frac{\lambda}{(1 + \eta_o)^k} = \frac{S^k \lambda}{(S^k (1 + \eta_o)^k)} \geq \frac{\varepsilon}{(1 + \eta(\theta))^k}$$

with $\varepsilon = S^k \lambda$. To interpret this result, consider the hypersurface $\xi = \frac{\varepsilon}{(1 + \eta)^k}$. It divides the phase space in two parts:

the interior part $S_- = \{z \mid \xi < \frac{\varepsilon}{(1 + \eta)^k}\}$

the exterior part $S_+ = \{z \mid \xi > \frac{\varepsilon}{(1 + \eta)^k}\}$.

Our result tells us that at the end of each evasion process $z(\theta) \in S_+$.

If we start outside (i.e., in S_+), then as long as we are in S_+, we don't worry. When we reach the hypersurface S we turn on the evasion control and we take this time as the origin of time. Then $z_o \in S$, i.e., $\xi_o = \dfrac{\varepsilon}{(1+\eta_o)^k}$.

By inequality (2) we have

$$\xi(t) \geq \frac{c_o \varepsilon^k}{(1+\eta_o)^{k^2+2k-1}}$$

and by the same argument which was used to get (3),

$$\xi(t) \geq \frac{c\varepsilon^k}{(1+\eta(t))^{k^2+2k-1}} .$$

Let us introduce the surface S': $\xi = \dfrac{c\varepsilon^k}{(1+\eta(t))^{k^2+2k-1}}$

where S'_+ and S'_- are the exterior and the interior of this surface S'. So during all the time we apply this evasion control we know that our point remains in S'_+; at the end of this interval of time we have (1) again and so our point is back in S_+, and we can do the same maneuver again.

Now what happens if $z_o \in S_-$. Then, during the first interval of time $[0,\theta]$ we have, from (2),

$$\xi(t) > \frac{c_o \xi_o^k}{(1+\eta_o)^{2k-1}} \geq \frac{c'\xi_o^k}{(1+\eta(t))^{2k-1}} .$$

This means that there is no capture between time 0 and time θ, and that at time θ we are back in S_+, and we can act as in the first case.

These results are more precise than the one stated in the first lecture. We can resume them in the following result, which is weaker and true only if $\xi_o \leq \varepsilon$,

$$\xi(t) > \frac{c_o \xi_o^k}{(1 + \eta(t))^{k^2+2k-1}} .$$

But the proof is not yet completed: there remains to check that the sum of the different intervals of time θ is infinite, in other words, that we have considered the problem for every $t \geq 0$.

θ is defined by $\theta = \frac{\theta_o}{1 + \eta_o}$, where θ is fixed.

If η_o remains finite, then θ is bounded from below and the series $\sum \theta$ diverges.

If η_o becomes infinite, then as η ist at most linear in θ, the series still diverges.

The proof is yet completed.

Comments. The solution was based on the maneuver superiority of the evader, but the pursuer can have a speed superiority. If at the beginning of the game they stand very far from each other, the evader cannot prevent the distance to decrease and the evader to pass very near from him, but then the distance will increase again.

It is not clear why our estimates are bad when η is big. In the example where η represents the velocities, this seems intuitive.

Lecture 9

The Pursuit Problem

General remarks on the non-linear case

As previously, we have a euclidean vector space R, and a vector z, $z \in R$, whose motion is given by $\dot{z} = F(z,u,v)$. The game is completed when $z \in M \subset R$, M is a given set in R.

Rules: Three types of rules can be used, depending on the available information at time t to choose $u(t)$.

1) The state only is known: $u = u(z)$.

2) The state and the opponent's control are known: $u = u(z,v)$, where $v = v(t)$.

3) The state is known, as well at the opponent's control history for a short interval of time in the future: $u = u[z,v(s)]$, where $t \leq s \leq t+\varepsilon$, $\varepsilon > 0$.

The third rule has also a practical meaning. We shall use it later.

Questions: For any of these three different games there are three questions to answer:

1) Starting from a given z_o, is it possible to complete the game?

2) Find the time $T(z_o)$ which is sufficient to do so. This number T will be called the estimating number, or, regarding it as a function $T(z)$, the estimating function. It is the interesting thing to find.

3) Is this $T(z)$ the best possible estimating function?
If there is no positive number δ such that $T-\delta$ is also an estimating number, then T is said to be optimal.

This optimality means that it is possible to find a control history $v(t)$ such that the pursuit will last a time arbitrarily close to T. It must be understood that this history can depend on everything else: u, the past history of the game, ... If there exists an evasion rule such that the process lasts exactly the time $T(z_o)$, then this function is said to be a "strong optimal" estimating function.

Thus we have defined three concepts: the estimating function, the optimal estimating function and the strong optimal. Thereafter we shall consider only the first one.

Construction of the function $T(z)$

Given the evader's control from zero to ε as

$$v = \hat{v}(t) , \qquad 0 \leq t \leq \varepsilon ,$$

Consider some control $u(t)$ for the pursuer.

We place these control histories in the differential equation of the game:

$$\dot{z} = F(z, u(t), \hat{v}(t)) , \qquad \text{with} \quad z(0) = z_o .$$

This gives a solution which we call $z(t)$ and, in particular, this gives a $z(\varepsilon) = z_1$ which depends on the $u(t)$ we have chosen.

Now consider the values $T(z_o)$ and $T(z_1)$ of the function $T(z)$. The difference $T(z_o)-T(z_1)$ depends on the history $u(t)$. Among all these possible histories, choose the one that maximizes this difference. Call it $\hat{u}(t)$

$$\hat{u}(t) = \arg \max_{u(\cdot)} [T(z_o) - T(z_1)] .$$

If for this function $\hat{u}(t)$

$$T(z_o) - T(z_1) \geq \varepsilon ,$$

then $T(z)$ is an estimating function. This also gives the way to construct the best possible u.

Comment on Isaacs' work[1]

Isaacs assumes that T is continuously differentiable, and that it is a strong optimum. Then letting ε go to zero, our relation becomes his "main equation". But there are cases where T does not fulfill these conditions.

Remark: In the case where the third rule is used (v is known a time ε in the future), the function $T(z)$ we are looking for might depend on ε. We require that the function we find be independent of ε and hold for every positive ε. For the linear case we are going to construct the function $T(z)$ a priori, without the help of an equation, and then show that it is an estimating function.

Linear case.

We consider now the case where the differential equation of the game has the form

$$\dot{z} = cz + v - u ,$$

where $v \in V$, $u \in U$. V and U are compact convex subsets of R. Completion is obtained when $z \in M$ arbitrary convex closed set.

[1] R. Isaacs: Differential Games, John Wiley and Sons, 1965.

We shall now construct , for positive τ, a set M_τ such that $M_o = M$; and M_τ is a continuous function of τ and considering, for a given z, the expression $e^{\tau z} z$, we focus our attention on the inclusion (1); verified or not

$$e^{\tau c} z \in M_\tau \,. \tag{1}$$

We shall prove that $T(z) = \tau_o$, where τ_o is the smallest τ such that (1) is true.

Construction of M_τ. Particular case: M is a vectorial subspace. As previously, L is the orthogonal complement of M, π is the orthogonal projection on L. We shall construct a subset $W_\tau \subset L$ and define

$$M_\tau = \{z: \ \pi z \in W_\tau\} \qquad \text{with } W_o = \{0\}$$

so that relation (1) is equivalent to

$$\pi e^{\tau c} z \in W_\tau \,.$$

Definitions: Let us define several concepts in set theory.

Generalized addition: Given two convex subsets $A, B \subset L$ and two numbers α and β, define

$$D = \alpha A + \beta B \qquad \text{by} \quad D \triangleq \{\alpha x + \beta y : x \in A, y \in B\} \,.$$

It is clear that D is convex. If A and B are compact, D is compact. Notice that for $\alpha = \beta = 1$ we have the classical sum and for $\alpha = -\beta = 1$ we have the algebraic difference.

Distance between two sets: Let H_r denote a ball of center at zero and radius r:

$$H_z \triangleq \{x: \|x\| \leq r\}$$

and consider the two inclusions, always verified for some large enough r,

$$A \subset B + H_r$$

$$B \subset A + H_r .$$

We call distance between the sets A and B the smallest r such that these two inclusions hold.

Integral: L being a finite dimensional euclidean vector space, and K the set of all compact subsets of L. (But $\emptyset \notin K$) K is not a vector space, as no set except from {0} seems to have an additive inverse. It could probably be imbedded into one by considering formal differences

$$A - B, \qquad A,B \in K.$$

The fact that this is a Banach space should be checked. It is complete. Thus given a continuous family A_τ, $A_\tau \in K \ \forall \tau$, one can define the integral

$$E = \int_0^t A_\tau d\tau .$$

Theorem: It is known that, given any $y \in E$, there exists a family $x(\tau)$ such that $x(\tau) \in A_\tau \ \forall \tau$, and

$$y = \int_0^t x(\tau)\, d\tau .$$

Now if y belongs to the boundary E' of E, then the corresponding $x(\tau)$ belong to the boundary A'_τ of A_τ for almost all τ.

Geometric subtraction: Given A and B, two subsets belonging to K, define

$$A \dot{-} B = D \qquad \text{by} \qquad D \triangleq \{x | x + B \subset A\} .$$

If this set is not empty, then it is compact and convex. D is a function of A and B, $D(A,B)$, but it is not defined for all A and B. Notice that by its definition $D = A \dot{-} B$ is such that

$$D + B \subset A .$$

Continuity: The question arises of knowing if this function $D(A,B)$ is continuous. The answer is that given A_o and B_o if

$$\dim(A_o \dot{-} B_o) = \dim L = \nu ,$$

then the function $D(A,B)$ is continuous at A_o, B_o.

Lecture 10

We shall use the operations previously defined to find the estimating function in the case where

$$\dot{z} = cz + v - u \qquad \left| \begin{matrix} u \in P \\ v \in Q \end{matrix} \right. \text{ convex compact subsets of } R$$

Completion for $z \in M$ $\quad$ M vectorial subspace of R; L orthogonal complement of M

π is the projection from R onto L, and as previously,

$$P_\tau \triangleq \pi e^{\tau c} P, \qquad Q_\tau \triangleq \pi e^{\tau c} Q.$$

Condition for capture: In the first lecture, a condition had been given under which capture can occur. We give now another form of the same condition. Let S_τ be

$$S_\tau = P_\tau \dot{-} Q_\tau.$$

Our condition is dim $\quad = \nu$ $\forall$ small $\tau > 0$, or more formally, for all τ: $0 < \tau < \hat{\tau}$ for some $\hat{\tau}$, possibly infinite, and we shall always consider a τ belonging to this interval.

We construct the set W_τ, defining it as

$$W_\tau = \int_0^\tau S_r \, dr$$

The estimating function is now defined by means of this set. Consider the inclusion (1), which can be true or not,

(1) $$\pi e^{\tau c} z_o \in W_\tau$$

and call τ_o the smallest τ such that it is verified. We claim that $T(z_o) = \tau_o$.

Remark: If $\tau_o = 0$, this means that

$$\pi z_o \in W_o = \{0\} .$$

This means that z belongs to M, the game is finished, which is consistent with the result $T(z_o) = 0$.

Proof of our claim: We shall first consider the second term:

$$W_\tau = \int_0^\tau S_r dr = \int_0^{\tau-\varepsilon} S_r dr + \int_{\tau-\varepsilon}^\tau S_r dr = W_{\tau-\varepsilon} + \int_{\tau-\varepsilon}^\tau S_r dr ,$$

provided that $\tau-\varepsilon \geq 0$.

Let us now look at the last integral:

$$S_r = P_r \underline{*} Q_r = \pi e^{rc} P \underline{*} \pi e^{rc} Q$$

implies, as seen earlier,

$$S_r + \pi e^{rc} Q \subset \pi e^{rc} P.$$

It is easy to see that such an inclusion can be integrated,

$$\int_{\tau-\varepsilon}^\tau (S_r + \pi e^{rc} Q) dr \subset \int_{\tau-\varepsilon}^\tau \pi e^{rc} P \, dr .$$

From the definition of the algebraic sum it is clear that the inclusion still holds if one of the two sets on the left side is replaced by one of its elements.

Notice that $v(t)$ is known for $0 < t < \varepsilon$, so that $v(\tau-r)$ is known for $\tau-\varepsilon < r < \tau$ and belongs to Q. So we can write

$$(2) \qquad \int_{\tau-\varepsilon}^{\tau} S_r\,dr + \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} v(\tau-r)dr \subset \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} P\,dr .$$

Let us turn now to the relation (1), $\pi e^{rc} z_o \in W_\tau$. We can write it as follows:

$$\pi e^{\tau c} z_o + \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} v(\tau-r)dr \in W_{\tau-\varepsilon} + \int_{\tau-\varepsilon}^{\tau} S_r\,dr + \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} v(\tau-r)dr ,$$

and making use of (2), we have

$$(3) \qquad \pi e^{\tau c} z_o + \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} v(\tau-r)dr \in W_{\tau-\varepsilon} + \int_{\tau-\varepsilon}^{\tau} \pi e^{rc} P\,dr .$$

This relation is verified for $\tau = \tau_o$; but it might be true for some smaller τ. Let us call τ_1 the smallest τ such that (3) is true, $\tau_1 \leq \tau_o$,

$$\pi e^{\tau_1 c} z_o + \int_{\tau_1-\varepsilon}^{\tau_1} \pi e^{rc} v(\tau_1-r)dr \in W_{\tau_1-\varepsilon} + \int_{\tau_1-\varepsilon}^{\tau_1} \pi e^{rc} P\,dr .$$

From relation (3) it follows that there exists a function $u(\tau_1-r) \in P$ defined on the interval $[0,\varepsilon]$ and such that $u(\tau_1-r) \in P$, $\forall r \in [\tau_1-\varepsilon, \tau_1]$ and such that

$$\pi e^{\tau_1 c} z_o + \int_{\tau_1-\varepsilon}^{\tau_1} \pi r^{rc} v(\tau_1-r)dr \in W_{\tau_1-\varepsilon} + \int_{\tau_1-\varepsilon}^{\tau_1} \pi e^{rc} u(\tau_1-r)dr .$$

$u(\tau_1 - r)$ is not any element of P, but it is defined by this relation. This last relation we rewrite as

$$\pi e^{\tau_1 c} z_o + \int_{\tau_1-\varepsilon}^{\tau_1} \pi e^{rc}[v(\tau_1 - r) - u(\tau_1 - r)]dr \in W_{\tau_1-\varepsilon},$$

and we modify its left side in

$$\pi e^{(\tau_1-\varepsilon)c}\{e^{\varepsilon c} z_o + \int_{\tau_1-\varepsilon}^{\tau_1} e^{[r-(\tau_1-\varepsilon)]c}[v(\tau_1 - r) - u(\tau_1 - r)]dr\}.$$

Then we make the change of variable of integration $s = r - (\tau_1 - \varepsilon)$. This same term becomes

$$\pi e^{(\tau_1-\varepsilon)c}\{e^{\varepsilon c} z_o + \int_{o}^{\varepsilon} e^{sc}[v(\varepsilon - s) - u(\varepsilon - s)]ds\} = \pi e^{(\tau_1-\varepsilon)c} z_1,$$

where z_1 is by definition the quantity between braces. It is clear that z_1 is also the value $z(\varepsilon)$ of the solution $z(t)$ of the differential equation of the game, with initial value $z(0) = z_o$ and applying the controls $v(t)$, which was given, and $u(t)$ that we found. Our result is

$$\pi e^{(\tau_1-\varepsilon)c} z_1 \in W_{\tau_1-\varepsilon}.$$

This relation means, with our definition of $T(z)$,

$$T(z_1) \leq \tau_1 .$$

Because τ_1 is the smallest τ for which relation (3) holds, it is possible to see that actually

$$T(z_1) = \tau_1 .$$

We now check the relation that a function $T(z)$ must verify to be an estimating function

$$T(z_o) - T(z_1) = \tau_o - \tau_1 + \varepsilon,$$

and because as we have seen, $\tau_1 \leq \tau_o$,

$$T(z_o) - T(z_1) \geq \varepsilon.$$

So $T(z)$ is an estimating function.

It can happen that the difference will actually be smaller than ε.

Remark: Our function $T(z)$ is defined independently of ε, and the result holds for all $\varepsilon > 0$.

It would be intersting to find what happens for ε going to zero. It might be possible to find a u according to the second rule instead of the third, namely, with the knowledge of current state and opponent's control, but not of its control in the future. May be we could find some bound on ε. This question is not solved yet.

Lecture 11

Before giving an example of use of the previous theory, let us make some remarks about the geometric subtraction.

Remark: As previously, let H_r be a ball of radius r and center 0. It is easily seen that

$$H_r \stackrel{*}{-} H_s = H_{r-s} \qquad \text{if} \quad s \leq r .$$

Given a function $r(\tau)$, real parameter τ, it is easily seen that

$$\int_0^t H_{r(\tau)}\, d\tau = H_{\hat{r}(t)}$$

with

$$\hat{r}(t) = \int_0^t r(\tau)d\tau .$$

Example of pursuit process.

We consider the example used in the evasion process: In a euclidean space E of dimension $\nu \geq 2$, a point x (the pursuer) and a point y (the evader) vary according to the dynamic possibilities:

$$\ddot{x} + \alpha\dot{x} = a \qquad \|a\| \leq \rho$$

$$\ddot{y} + \beta\dot{y} = b \qquad \|b\| \leq \sigma ,$$

α, β, ρ and σ are positive numbers. The game is completed when $x = y$.

We put this description in the form of a differential game, as was done earlier, defining the state $z \in R$,

$$z = (z^1, z^2, z^3)$$

$$\left.\begin{aligned} z^1 &= x-y \\ z^2 &= \dot{x} \\ z^3 &= \dot{y} \end{aligned}\right\} \quad z^1,\ z^2 \text{ and } z^3 \text{ are components (not coordinates) of } z.$$

The subspaces M and L are defined by

$$M = \{(0,z^2,z^3)\}\ , \qquad L = \{(z^1,0,0)\}\ .$$

There is an obvious isomorphism between L and E, and L will be identified with E thereafter. The projection π will be considered as a mapping of R on E.

We have already shown that

$$\pi(z^1,z^2,z^3) = z^1 \qquad (z^1 \in E, \text{ not } L).$$

The operator $\pi e^{\tau C}$, first line of the matrix $e^{\tau C}$, is

$$\pi e^{\tau C} = (1 \quad f(\tau) \quad -g(\tau))$$

$$\pi e^{\tau C} z_o = \pi e^{\tau C}(z_o^1,z_o^2,z_o^3) = z_o^1 + z_o^2 f(\tau) - z_o^3 g(\tau)$$

and

$$P_\tau = \{x \mid \|x\| \leq \rho f(\tau)\}\ , \qquad Q_\tau = \{x \mid \|x\| \leq \sigma g(\tau)\}\ ,$$

where the two functions $f(\tau)$ and $g(\tau)$ are

$$f(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha}\ , \qquad g(\tau) = \frac{1 - e^{-\beta\tau}}{\beta}$$

Remarks about the functions f and g:

$$f(0) = g(0) = 0$$

$$\dot{f}(\tau) = e^{-\alpha\tau} \qquad \dot{g}(\tau) = e^{-\beta\tau}$$

$$\dot{f}(0) = \dot{g}(0) = 1 .$$

We will also need their limit for $\tau \to \infty$,

$$\lim_{\tau \to \infty} f(\tau) = \frac{1}{\alpha} , \qquad \lim_{\tau \to \infty} g(\tau) = \frac{1}{\beta} .$$

Geometrical difference

$$S_\tau = P_\tau \stackrel{*}{-} Q_\tau = \{x \mid \|x\| \leq h(\tau)\} = H_{h(\tau)} ,$$

where $h(\tau) = \rho f(\tau) - \sigma g(\tau)$.

Because the difference is not defined for $h(\tau) < 0$, we must look at this function.

$$h(0) = 0 , \qquad \dot{h}(0) = \rho - \sigma > 0 ,$$

since, by assumption, we are in the case where the pursuer has maneuvering superiority over the evader: $\rho > \sigma$

$$\dot{h}(\tau) = \rho\, e^{-\alpha\tau} - \sigma\, e^{-\beta\tau}$$

$$\dot{h}(\tau) = 0 \qquad \text{for} \quad e^{(\alpha-\beta)\tau} = \frac{\rho}{\sigma} \qquad \underline{\text{one point at most}}$$

and looking at the limit

$$\lim_{\tau \to \infty} h(\tau) = \frac{\rho}{\alpha} - \frac{\sigma}{\beta} ;$$

we distinguish two cases,

$\frac{\rho}{\alpha} - \frac{\sigma}{\beta} > 0$: $h(\tau)$ is always positive and greater than some positive constant for τ greater than some positive value.

$\frac{\rho}{\alpha} - \frac{\sigma}{\beta} < 0$: There exists a unique positive τ such that $h(\hat{\tau}) = 0$ and $h(\tau) > 0$ for $0 < \tau < \hat{\tau}$.

Set W_τ

We have

$$W_\tau = \int_0^\tau S_r \, dr = H_{\hat{h}(\tau)} \qquad \text{a sphere,}$$

where

$$\hat{h}(\tau) = \int_0^\tau h(r) dr .$$

Estimating function

The inclusion $\pi e^{\pi z_o} \in W_\tau$ can now be written as an analytical expression since both sides are known

$$\left\| z_o^1 + f(\tau) z_o^2 - g(\tau) z_o^3 \right\|^2 \leq \hat{h}^2(\tau) ,$$

and τ_o is the smallest τ for which this inequality holds. By continuity it is clear that for τ_o we shall have the exact equality

$$\left\| z_o^1 + f(\tau_o) z_o^2 - g(\tau_o) z_o^3 \right\|^2 = \hat{h}^2(\tau_o) , \qquad \tau_o > 0 .$$

And $T(z_o) = \tau_o$, τ_o being the smallest root of the above equation.

Remark: Any of the roots of this equation defines a function $T(z)$ which verifies Isaacs' "main equation" (Partial differential equation). But then it has several values, and insisting on choosing the lowest positive one may lead to discontinuities in T and its derivatives.

So the hypotheses on which Isaacs' theory was built do not hold here. Moreover, we have an explicit solution instead of a partial differential equation.

Let us look at the case where the relative value of the limit velocities is,

$$\frac{\rho}{\alpha} > \frac{\sigma}{\beta} .$$

Then $h(\tau) \to \infty$, where $\tau \to \infty$, so that, whatever be z_o, there exists a τ_o positive such that the equality is verified. So the game shall always be completed, capture will occur for any starting position.

It has been proved by Nikolsky that for a rather restrictive class of games, containing this one, this estimating function is also optimal.

Remarks: We know from the general theory developed earlier that the function $T(z)$ we have computed here is actually an estimating function. We have also given the construction of the best possible u in the general case. In this case we can give a different, but very simple, construction of u which ensures capture in exactly $T(z_o)$. This construction will not make use of the knowledge of v at a time greater than current time. But it will also not take advantage of the possible "mistakes" of the evader.

Consider the inclusion

$$\pi\, e^{\tau_o c}\, z_o \in \int_o^{\tau_o} S_r\, dr$$

means that there exists a measurable vector function $s(r)$ such that

$$\pi\, e^{\tau_o c}\, z_o = \int_o^{\tau_o} s(r)dr \qquad\qquad s(r) \in S_r \quad \forall r\,.$$

Now the definition of S is such that

$$S_r + \pi\, e^{rc} Q \subset \pi e^{rc} P\,,$$

which means

$$s(r) + \pi\, e^{rc} v \in \pi e^{rc} P\,, \qquad \forall s_r \in S_r\,, \quad \forall\, v \in Q\,,$$

so for any given z_o ($\Longrightarrow$ given $s(r)$) and any given v there exists a $u \in P$ such that

$$s(r) + \pi\, e^{rc} v = \pi e^{rc} u\,.$$

If in this equality we put for v the evader's control $v(\tau_o - r)$, then it gives us a pursuit control $u(\tau_o - r)$. Thus we have

$$s(r) + \pi e^{rc}[v(\tau_o - r) - u(\tau_o - r)] = 0 \qquad \forall r$$

and integrating from zero to τ_o, taking π out of the integral,

$$\pi\{e^{\tau_o c}\, z_o + \int_o^{\tau_o} e^{rc}[v(\tau_o - r) - u(\tau_o - r)]dr\} = 0\,,$$

which means that $z(\tau_o) \in M$. Q.E.D.

Note that this <u>does not</u> take advantage of possible mistakes of the evader.